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Prolongation structures and Bäcklund transformations for the matrix Korteweg–de Vries and the Boomeron equation

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Abstract. Prolongation structures are determined for the matrix Korteweg-de Vries and the Boomeron equation by using the integrability condition for a linear system of first-order equations. Symmetries of these prolongation structures are used to derive Bäcklund transformations and to construct solutions for both equations.

1. Introduction

It is well known that the KdV equation $u_t + \alpha u u_x + \beta u_{xxx} = 0$ does possess the Lie algebra $SL(2, \mathbb{R})$ as a prolongation structure. This was first discovered by Wahlquist and Estabrook (1973), who obtained this prolongation structure, by using the theory of exterior differential systems, as a subalgebra of a more general prolongation. Chern-Peng (1979) discovered that this prolongation structure followed in a natural way from the Maurer-Cartan equations of the $SL(2, \mathbb{R})$ algebra. From the $SL(2, \mathbb{R})$-prolongation structure it is possible to derive a Bäcklund transformation and thus a method for constructing solutions for the KdV equation.

The Bäcklund transformation and the explicit construction of solutions can, as we shall show, easily be generalised to systems of equations, i.e. equations in matrix form. In this paper we restrict ourselves to the KdV equation in matrix form and to the Boomeron equation, but it will be clear that our method can be used for other systems of equations as well. The matrix KdV equation has been used in investigations concerning Jupiter's red spot (Redekopp *et al* 1978); furthermore the components may be used to construct solutions of the scalar KdV equation or other equations of physical interest (Wadati 1980).

In order to make our generalisation transparent we treat in § 2 the scalar case; it will appear that a simplification of the construction of a well known Bäcklund transformation is possible. In § 3 we generalise the scalar case to systems of equations of the KdV type and in § 4 we treat generalisations of another type among which is the Boomeron equation.

2. The scalar case

2.1. *The linear prolongation structure SL(2, ℝ)*

We consider the system of equations in $\mathbb{R}^2$ with coordinates (x, t)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_x = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_t = B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.1}$$

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where

$$A=\begin{pmatrix}\lambda(x,t) & u(x,t)\\ r(x,t) & -\lambda(x,t)\end{pmatrix}, \qquad B=\begin{pmatrix}a(x,t) & b(x,t)\\ c(x,t) & -a(x,t)\end{pmatrix};$$

$y_1=y_1(x,t)$ and $y_2=y_2(x,t)$ and subscripts denote differentiation.

The compatibility condition: $\binom{y_1}{y_2}_{xt}=\binom{y_1}{y_2}_{tx}$ gives

$$-A_t+B_x=[A,B] \tag{2.2}$$

where $[A,B]=AB-BA$ is the commutator of A and B. (2.2) leads to

$$-\lambda_t+a_x=uc-\lambda b, \qquad -u_t+b_x=2\lambda b-2ua,$$
$$-r_t+c_x=-2\lambda c+2ra. \tag{2.3a, b, c}$$

With the special choice $\lambda(x,t)\equiv\lambda\equiv$ constant and $r(x,t)\equiv 1$ the set of equations (2.3) becomes

$$a_x=uc-b, \qquad u_t=2ua-2\lambda b+b_x, \qquad c_x=-2\lambda c+2a$$

or

$$a=\lambda c+\tfrac{1}{2}c_x \tag{2.4a}$$
$$b=uc-\lambda c_x-\tfrac{1}{2}c_{xx} \tag{2.4b}$$
$$u_t=2uc_x+u_xc+2\lambda^2c_x-\tfrac{1}{2}c_{xxx}. \tag{2.4c}$$

In order to eliminate λ from (2.4c) we choose $c=\lambda^2-\tfrac{1}{2}u$ which gives the KdV equation

$$u_t=-\tfrac{3}{2}uu_x+\tfrac{1}{4}u_{xxx}. \tag{2.5}$$

Finally substituting (2.4) into (2.1) we obtain the result that the system

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix}_x=\begin{pmatrix}\lambda & u\\ 1 & -\lambda\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$
$$\begin{pmatrix}y_1\\ y_2\end{pmatrix}_t=\begin{pmatrix}\lambda^3-\tfrac{1}{2}\lambda u-\tfrac{1}{4}u_x & \lambda^2u-\tfrac{1}{2}u^2+\tfrac{1}{2}\lambda u_x+\tfrac{1}{4}u_{xx}\\ \lambda^2-\tfrac{1}{2}u & -(\lambda^3-\tfrac{1}{2}\lambda u-\tfrac{1}{4}u_x)\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} \tag{2.6}$$

is completely integrable when u satisfies (2.5). In other words, (2.6) defines a linear prolongation of (2.5).

If one substitutes in (2.4c) for c a polynomial of degree n in λ^2, one obtains the result that the so-called KdV equation of order n provides an integrability condition for the pertaining system (2.1) (Chern-Peng 1979).

2.2. *Quadratic prolongation and Bäcklund transformation*

From (2.6) we can (at least locally) derive another prolongation involving only one potential by substituting $y=y_1y_2^{-1}$. This gives

$$y_x=-y^2+2\lambda y+u$$
$$y_t=(\tfrac{1}{2}u-\lambda^2)y^2+(2\lambda^3-\lambda u-\tfrac{1}{2}u_x)y+\lambda^2u-\tfrac{1}{2}u^2+\tfrac{1}{2}\lambda u_x+\tfrac{1}{4}u_{xx}. \tag{2.7}$$

From these equations we eliminate u, u_x and u_{xx} and we find that $y=y(x,t)$ satisfies

$$y_t=-\tfrac{3}{2}y^2y_x+\tfrac{1}{4}y_{xxx}+3\lambda yy_x \tag{2.8}$$

which is in fact a λ dependent modified KdV equation. Miura (1974) discovered that there is a close connection between the KdV and the modified KdV equation. We show that a similar relation holds also for the λ dependent modified KdV equation, by proving the next theorem.

Theorem 1. Suppose $y(x, t)$ is a solution of (2.8), then $u(x, t)$ defined by $u = y_x + y^2 - 2\lambda y$ satisfies the KdV equation (2.5).

Proof. If $u = y_x + y^2 - 2\lambda y$, then

$$u_t = y_{xt} + 2yy_t - 2\lambda y_t = (\partial/\partial x + 2y - 2\lambda)y_t$$

and by a direct calculation it follows from (2.8) and (2.7) that u satisfies (2.5).

This theorem together with the prolongation condition makes it possible to derive a Bäcklund transformation for the KdV equation. First, we notice that the substitution $y = z + \lambda$ in (2.8) gives an odd equation in z of the form:

$$z_t = -\tfrac{3}{2}z^2 z_x + \tfrac{1}{4}z_{xxx} + \tfrac{3}{2}\lambda^2 z_x. \tag{2.9}$$

Since (2.9) is odd it possesses with every solution z another solution $-z$, and from this it follows that (2.8) possesses with every solution y another solution $-y + 2\lambda$.

Now, let u be a solution of the KdV equation (2.5), then by the prolongation condition, (2.7) is completely integrable and y satisfies (2.8). But then $-y + 2\lambda$ also satisfies (2.8) and by using theorem 1 we find that $u' = u'(x, t)$ defined by

$$u' = (-y + 2\lambda)_x + (-y + 2\lambda)^2 - 2\lambda(-y + 2\lambda)$$

or

$$u' = -y_x + y^2 - 2\lambda y$$

is another solution of the KdV equation.

Substituting $u = y_x + y^2 - 2\lambda y$ we find a Bäcklund transformation

$$u' = -u + 2y^2 - 4\lambda y. \tag{2.10}$$

This, in fact, is the same Bäcklund transformation as that found by Wahlquist and Estabrook (1973); however, we were able to find it without, as they stated, a tedious calculation. Summarising we have obtained the theorem

Theorem 2. Whenever u is a solution of the KdV equation (2.5) and y a solution of the system (2.7), then

$$u' = -u + 2y^2 - 4\lambda y$$

is also a solution of the KdV equation (2.5).

Inserting $u \equiv 0$ yields for real values of λ: $y - \lambda = \lambda \tanh \lambda(x + \lambda^2 t + \phi_0)$ or $y - \lambda = \lambda \coth \lambda(x + \lambda^2 t + \phi_0)$, where ϕ_0 is an integration constant. Using finally (2.10) we obtain the well known stationary solutions

$$u' = -2\lambda^2[\operatorname{sech} \lambda(x + \lambda^2 t + \phi_0)]^2 \tag{2.11}$$

resp.

$$u' = +2\lambda^2[\operatorname{cosech} \lambda(x + \lambda^2 t + \phi_0)]^2. \tag{2.12}$$

3. The matrix case

3.1. Linear prolongation structure for the matrix KdV equation

We consider the system

$$\begin{pmatrix} y \\ z \end{pmatrix}_x = \left(\begin{array}{c|c} \lambda I & U \\ \hline R & -\lambda I \end{array}\right)\begin{pmatrix} y \\ z \end{pmatrix} = A\begin{pmatrix} y \\ z \end{pmatrix}$$
$$\begin{pmatrix} y \\ z \end{pmatrix}_t = \left(\begin{array}{c|c} a & b \\ \hline c & -a \end{array}\right)\begin{pmatrix} y \\ z \end{pmatrix} = B\begin{pmatrix} y \\ z \end{pmatrix} \tag{3.1}$$

where

$$A = \left(\begin{array}{c|c} \lambda I & U \\ \hline R & -\lambda I \end{array}\right), \qquad B = \left(\begin{array}{c|c} a & b \\ \hline c & -a \end{array}\right),$$

U, R, a, b, c, y, z are 2×2 matrix functions defined on $\mathbb{R}^2$ with coordinates (x, t) and I is the unit 2×2 matrix and λ a constant. It is clear that (3.1) is a generalisation of the system (2.1) with $\lambda(x, t) \equiv \lambda \equiv$ constant.

As in the scalar case the compatibility condition

$$\begin{pmatrix} y \\ z \end{pmatrix}_{xt} = \begin{pmatrix} y \\ z \end{pmatrix}_{tx}$$

gives

$$B_x - A_t = [A, B]. \tag{3.2}$$

Choosing $R \equiv I$ we obtain

$$B_x = \left(\begin{array}{c|c} a_x & b_x \\ \hline c_x & -a_x \end{array}\right), \qquad A_t = \left(\begin{array}{c|c} 0 & U_t \\ \hline 0 & 0 \end{array}\right)$$

and (3.3)

$$[A, B] = \left(\begin{array}{c|c} Uc - b & 2\lambda b - Ua - aU \\ \hline 2a - 2\lambda c & b - cU \end{array}\right).$$

Therefore we obtain for the compatibility condition the set of equations

$$a_x = Uc - b, \qquad b_x - U_t = 2\lambda b - Ua - aU, \tag{3.4a, b}$$
$$c_x = 2a - 2\lambda c, \qquad -a_x = b - cU. \tag{3.4c, d}$$

From (3.4a) and (3.4d) we obtain $Uc = cU$ so $[c, U] = 0$ and we write (3.4) as

$$a = \lambda c + \tfrac{1}{2}c_x, \qquad b = Uc - \lambda c_x - \tfrac{1}{2}c_{xx}, \tag{3.5a, b}$$
$$U_t = U_x c + \tfrac{3}{2}Uc_x + \tfrac{1}{2}c_x U - \tfrac{1}{2}c_{xxx} + 2\lambda^2 c_x. \tag{3.5c}$$

Making the special choice $c = \lambda^2 I - \tfrac{1}{2}U$ we are able to eliminate λ^2 from (3.5c) and we obtain for U the matrix KdV equation

$$U_t = -\tfrac{3}{4}\{U, U_x\} + \tfrac{1}{4}U_{xxx} \tag{3.6}$$

where $\{U, U_x\} = UU_x + U_x U$ is the anti-commutator of U and U_x. Substitution of (3.6) with the special choice $c=\lambda^2 I-\frac{1}{2}U$ into (3.1) gives:

$$\begin{pmatrix} y \\ z \end{pmatrix}_x = \left(\begin{array}{c|c} \lambda I & U \\ \hline I & -\lambda I \end{array}\right)\begin{pmatrix} y \\ z \end{pmatrix}$$
$$\begin{pmatrix} y \\ z \end{pmatrix}_t = \left(\begin{array}{c|c} \lambda^3 I-\frac{1}{2}\lambda U-\frac{1}{4}U_x & \lambda^2 U-\frac{1}{2}U^2+\frac{1}{2}\lambda U_x+\frac{1}{4}U_{xx} \\ \hline \lambda^2 I-\frac{1}{2}U & -(\lambda^3 I-\frac{1}{2}\lambda U-\frac{1}{4}U_x) \end{array}\right)\begin{pmatrix} y \\ z \end{pmatrix}. \tag{3.7}$$

So (3.7) is completely integrable when U satisfies (3.6), i.e. (3.7) defines a linear prolongation of (3.6).

Remark. As in the scalar case we may take for the matrix C the choice

$$C=\sum_{j=0}^{n} C_j(x,t)\lambda^{2j}.$$

Substitution into (3.5c) yields a recurrent system for the coefficients $C_j(x,t)$ and one obtains finally a matrix KdV equation of the nth order. The latter equation will not be investigated in this paper.

3.2. Quadratic prolongation and Bäcklund transformation

As in the scalar case we can (at least locally) derive another prolongation from (3.7) by putting $\tilde{y}=yz^{-1}$. Substitution of $\tilde{y}=yz^{-1}$ into (3.7) yields

$$\begin{aligned} \tilde{y}_x &= -\tilde{y}^2+2\lambda\tilde{y}+U \\ \tilde{y}_t &= -\lambda^2\tilde{y}^2+\tfrac{1}{2}\tilde{y}U\tilde{y}+\tilde{y}(\lambda^3 I-\tfrac{1}{2}\lambda U-\tfrac{1}{4}U_x) \\ &\quad +(\lambda^3 I-\tfrac{1}{2}\lambda U-\tfrac{1}{4}U_x)\tilde{y}+\lambda^2 U-\tfrac{1}{2}U^2+\tfrac{1}{2}\lambda U_x+\tfrac{1}{4}U_{xx}. \end{aligned} \tag{3.8}$$

From this we eliminate U, U_x and U_{xx} and we find that $\tilde{y}$ satisfies a λ dependent modified KdV equation in matrix form

$$\tilde{y}_t=-\tfrac{3}{4}\{\tilde{y}^2,\tilde{y}_x\}+\tfrac{1}{4}\tilde{y}_{xxx}+\tfrac{3}{2}\lambda\{\tilde{y},\tilde{y}_x\}. \tag{3.9}$$

Since (3.8) defines a prolongation of (3.6) we know that if U satisfies the KdV equation (3.6), then $\tilde{y}$ defined by (3.8) satisfies the modified KdV equation (3.9), which is a nice generalisation of (2.8).

To find a Bäcklund transformation as in the scalar case we will prove that the converse is also true.

Theorem 3. Suppose $\tilde{y}$ satisfies (3.9), then U defined by $U=\tilde{y}_x+\tilde{y}^2-2\lambda\tilde{y}$ satisfies the matrix KdV equation (3.6).

Proof. Putting $\tilde{y}=z+\lambda I$ we find that z satisfies an equation odd in z

$$z_t=-\tfrac{3}{4}\{z^2,z_x\}+\tfrac{1}{4}z_{xxx}+\tfrac{3}{2}\lambda^2 z_x =: Q(z).$$

From $U=z_x+z^2-\lambda^2 I$ it follows that

$$U_t=z_{xt}+zz_t+z_t z=Q(z)z+zQ(z)+(\partial/\partial x)Q(z)$$

and by the definition of $Q(z)$ it follows that

$$U_t=-\tfrac{3}{4}\{U,U_x\}+\tfrac{1}{4}U_{xxx}.$$

Suppose U is a solution of the matrix KdV equation (3.6), then $\tilde{y}$ defined by (3.8) is a solution of the modified KdV equation (3.9) and $-\tilde{y}+2\lambda I$ is another solution of (3.9) and so it follows from theorem 3 that

$$U'=(-\tilde{y}+2\lambda I)_x+(-\tilde{y}+2\lambda I)^2-2\lambda(-\tilde{y}+2\lambda I)$$

or $U'=-\tilde{y}_x+\tilde{y}^2-2\lambda\tilde{y}$ is another solution of the KdV equation. With the substitution $U=\tilde{y}_x+\tilde{y}^2-2\lambda\tilde{y}$ we get a Bäcklund transformation

$$U'=-U+2\tilde{y}^2-4\lambda\tilde{y}. \tag{3.10}$$

To summarise we have the following theorem

Theorem 4. Whenever U is a solution of the matrix KdV equation (3.6) and $\tilde{y}$ a solution of the system (3.8) then

$$U'=-U+2\tilde{y}^2-4\lambda\tilde{y}$$

is also a solution of the matrix KdV equation (3.6).

3.3. Construction of a solution for the matrix KdV equation

Since $U\equiv 0$ is a solution of (3.6), we can find a new solution by using the Bäcklund transformation (3.10)

$$U'=2\tilde{y}^2-4\lambda\tilde{y} \tag{3.11}$$

where $\tilde{y}$ satisfies (3.8) with $U\equiv 0$, i.e.

$$\tilde{y}_x=-\tilde{y}^2+2\lambda\tilde{y}, \qquad \tilde{y}_t=-\lambda^2\tilde{y}^2+2\lambda^3\tilde{y}. \tag{3.12}$$

We see that $\tilde{y}_t=\lambda^2\tilde{y}_x$ so $\tilde{y}=\tilde{y}(x+\lambda^2 t)$.

Defining $\phi(\xi)=\tilde{y}-\lambda I$ with $\xi=x+\lambda^2 t$ we get for $\phi(\xi)$ the first-order equation

$$\phi_\xi=-\phi^2+\lambda^2 I. \tag{3.13}$$

(3.13) can be linearised by the substitution $\phi=\psi^{-1}\psi_\xi$ or $\psi_\xi=\psi\phi$ and we obtain:

$$\psi_{\xi\xi}=\psi_\xi\phi+\psi\phi_\xi=\psi\phi^2+\psi(-\phi^2+\lambda^2 I)$$

or

$$\psi_{\xi\xi}=\lambda^2\psi. \tag{3.14}$$

(3.14) is a set of uncoupled equations and the general solution reads

$$\psi=c^1\,\mathrm{e}^{\lambda\xi}+c^2\,\mathrm{e}^{-\lambda\xi}$$

with c^1 and c^2 arbitrary constant 2×2 matrices. However, since $\phi=\psi^{-1}\psi_\xi$ is invariant under the transformation $\psi\to A\psi$ with A a constant matrix, we may choose ψ as

$$\psi=I\,\mathrm{e}^{\lambda\xi}+C\,\mathrm{e}^{-\lambda\xi}$$

with

$$C=(c_{ij})=\begin{pmatrix} c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix} \qquad \text{and} \qquad I=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}.$$

From this result we compute the function ϕ and we get after a short calculation

$$\phi = \psi^{-1}\psi_\xi = \frac{1}{|\psi|}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\lambda\, e^{2\lambda\xi} - \begin{pmatrix}|c| & 0\\ 0 & |c|\end{pmatrix}\lambda\, e^{-2\lambda\xi} + \lambda\begin{pmatrix}c_{22}-c_{11} & -2c_{12}\\ -2c_{21} & c_{11}-c_{22}\end{pmatrix}\right]$$

with $|\psi| = \det\psi = e^{2\lambda\xi} + (\det c)\, e^{-2\lambda\xi} + (c_{11}+c_{22})$. Written explicitly, the result becomes

$$\phi_{11} = \lambda\left(\frac{e^{2\lambda\xi} - |c|\, e^{-2\lambda\xi} + (c_{22}-c_{11})}{e^{2\lambda\xi} + |c|\, e^{-2\lambda\xi} + (c_{22}+c_{11})}\right)$$
$$\phi_{12} = \frac{-2\lambda c_{12}}{e^{2\lambda\xi} + |c|\, e^{-2\lambda\xi} + (c_{22}+c_{11})} \tag{3.15}$$

$$\phi_{21} = \frac{-2\lambda c_{21}}{e^{2\lambda\xi} + |c|\, e^{-2\lambda\xi} + (c_{22}+c_{11})}$$
$$\phi_{22} = \lambda\left(\frac{e^{2\lambda\xi} - |c|\, e^{-2\lambda\xi} - (c_{22}-c_{11})}{e^{2\lambda\xi} + |c|\, e^{-2\lambda\xi} + (c_{22}+c_{11})}\right). \tag{3.16}$$

Using $U' = 2\tilde{y}^2 - 4\lambda\tilde{y} = 2(\phi+\lambda I)^2 - 4\lambda(\phi+\lambda I) = 2(\phi^2 - \lambda^2 I)$ we get after substitution of (3.15) and (3.16) finally a solution of the matrix KdV equation (3.6).

The result reads as follows

$$U'_{11} = -8\lambda^2\left(\frac{c_{11}\, e^{2\lambda\xi} + |c| c_{22}\, e^{-2\lambda\xi} + 2|c|}{[e^{2\lambda\xi} + |c|\, e^{-2\lambda\xi} + (c_{11}+c_{22})]^2}\right)$$
$$U'_{12} = -8\lambda^2 c_{12}\left(\frac{e^{2\lambda\xi} - |c|\, e^{-2\lambda\xi}}{[e^{2\lambda\xi} + |c|\, e^{-2\lambda\xi} + (c_{11}+c_{22})]^2}\right)$$
$$U'_{21} = -8\lambda^2 c_{21}\left(\frac{e^{2\lambda\xi} - |c|\, e^{-2\lambda\xi}}{[e^{2\lambda\xi} + |c|\, e^{-2\lambda\xi} + (c_{11}+c_{22})]^2}\right) \tag{3.17}$$
$$U'_{22} = -8\lambda^2\left(\frac{c_{22}\, e^{2\lambda\xi} + |c| c_{11}\, e^{-2\lambda\xi} + 2|c|}{[e^{2\lambda\xi} + |c|\, e^{-2\lambda\xi} + (c_{11}+c_{22})]^2}\right).$$

Hence (3.17) is a solution of the system of equations

$$U_t = -\tfrac{3}{4}\{U, U_x\} + \tfrac{1}{4}U_{xxx}$$

or written in components

$$U_{11t} = -\tfrac{3}{4}(U_{11}U_{11x} + U_{12}U_{21x} + U_{11x}U_{11} + U_{12x}U_{21}) + \tfrac{1}{4}U_{11xxx}$$
$$U_{12t} = -\tfrac{3}{4}(U_{11}U_{12x} + U_{12}U_{22x} + U_{11x}U_{12} + U_{12x}U_{22}) + \tfrac{1}{4}U_{12xxx}$$
$$U_{21t} = -\tfrac{3}{4}(U_{21}U_{11x} + U_{22}U_{21x} + U_{21x}U_{11} + U_{22x}U_{21}) + \tfrac{1}{4}U_{21xxx} \tag{3.18}$$
$$U_{22t} = -\tfrac{3}{4}(U_{21}U_{12x} + U_{22}U_{22x} + U_{21x}U_{12} + U_{22x}U_{22}) + \tfrac{1}{4}U_{22xxx}.$$

In order to check this result we consider the special solution of (3.18) with $U_{12} \equiv U_{21} \equiv 0$.

This choice implies that U_{11} and U_{22} satisfy the KdV equation and according to (3.17) we have $c_{12} = c_{21} = 0$. Inserting this into (3.17) we obtain

$$U'_{11} = -8\lambda^2\left(\frac{c_{11}\, e^{2\lambda\xi} + c_{11}c_{22}^2\, e^{-2\lambda\xi} + 2c_{11}c_{22}}{[e^{2\lambda\xi} + c_{11}c_{22}\, e^{-2\lambda\xi} + (c_{11}+c_{22})]^2}\right)$$

or

$$U'_{11} = -8\lambda^2 c_{11}\left(\frac{1}{(e^{\lambda\xi} + c_{11}\,e^{-\lambda\xi})^2}\right) \tag{3.19}$$

and in a completely similar way

$$U_{22} = -8\lambda^2 c_{22}\left(\frac{1}{(e^{\lambda\xi} + c_{22}\,e^{-\lambda\xi})^2}\right). \tag{3.20}$$

This solution yields for $c_{11} = \exp(-2\lambda\alpha_1)$ and $c_{22} = \exp(-2\lambda\alpha_2)$ the well known soliton solutions

$$U_{11} = \frac{-2\lambda^2}{\cosh^2[\lambda(x+\lambda^2 t+\alpha_1)]}, \qquad U_{22} = \frac{-2\lambda^2}{\cosh^2[\lambda(x+\lambda^2 t+\alpha_2)]}.$$

(Compare (2.11)-(2.12).)

4. The Boomeron equation

4.1. A general linear prolongation structure

The scheme of § 3 is easily generalised by considering instead of (3.1) the system

$$\begin{pmatrix} y \\ z \end{pmatrix}_x = \left(\begin{array}{c|c} A & B \\ \hline C & -A \end{array}\right) = \tilde{A}\begin{pmatrix} y \\ z \end{pmatrix}$$
$$\begin{pmatrix} y \\ z \end{pmatrix}_t = \left(\begin{array}{c|c} a & b \\ \hline c & -a \end{array}\right) = \tilde{B}\begin{pmatrix} y \\ z \end{pmatrix} \tag{4.1}$$

where A, B, C, a, b, c, y, z are 2×2 matrix functions defined on $\mathbb{R}^2$ with coordinates (x, t).

The compatibility condition gives again $\tilde{B}_x - \tilde{A}_t = [\tilde{A}, \tilde{B}]$ with

$$\tilde{B}_x = \left(\begin{array}{c|c} a_x & b_x \\ \hline c_x & -a_x \end{array}\right), \qquad \tilde{A}_t = \left(\begin{array}{c|c} A_t & B_t \\ \hline C_t & -A_t \end{array}\right)$$

and

$$[\tilde{A}, \tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A} = \left(\begin{array}{c|c} Aa - aA + Bc - bC & Ab + bA - Ba - aB \\ \hline Ca + aC - Ac - cA & Aa - aA + Cb - cB \end{array}\right)$$

or written explicitly

$$a_x - A_t = [A, a] + Bc - bC, \qquad b_x - B_t = \{A, b\} - \{B, a\}, \tag{4.2a, b}$$

$$c_x - C_t = \{C, a\} - \{A, c\}, \qquad -a_x + A_t = [A, a] + Cb - cB. \tag{4.2c, d}$$

Differentiating y_x again with respect to x yields

$$y_{xx} = (A_x + A^2 + BC)y + (B_x + [A, B])z. \tag{4.3}$$

Defining a matrix function U by

$$U = A_x + A^2 + BC + \lambda^2 I \tag{4.4}$$

and by making the special choice

$$B_x = -[A, B] \tag{4.5}$$

we obtain the result

$$y_{xx}=(U-\lambda^2 I)y. \tag{4.6}$$

Hence the matrix U is a potential for the matrix Schrödinger equation. It is quite natural to consider equations, that are solvable by the inverse scattering transformation, as have been treated by Calogero and Degasparis (1980).

In the rest of this paper we restrict ourselves to the Boomeron equation.

4.2. The linear prolongation structure of the Boomeron equation

In order to simplify equations (4.2) and (4.4)-(4.5) we make the assumptions

$$B\equiv I, \qquad C=-A^2-\lambda^2 I \tag{4.7}$$

and it follows that $U=A_x$ and (4.2) becomes

$$2[A, a]-[A^2, b]=0, \qquad A_t=a_x-c-\tfrac{1}{2}\{b, A^2\}-\lambda^2 b, \tag{4.8a, b}$$

$$2a=\{A, b\}, \qquad (A^2)_t=-c_x-\{A^2, a\}-2\lambda^2 a-\{c, A\}. \tag{4.8c, d}$$

Taking b as a constant matrix and after replacing b by $2b$ we get the following system, also used by Martini (1983)

$$[A, a]=[A^2, b], \qquad a=\{A, b\}, \tag{4.9a, b}$$

$$A_t=\{A_x, b\}-\{A^2, b\}-2\lambda^2 b-c, \tag{4.9c}$$

$$(A^2)_t=-c_x-\{c, A\}-\{A^2, a\}-2\lambda^2 a. \tag{4.9d}$$

We remark that (4.9a) follows from (4.9b) and using (4.9b), (4.9c) and (4.9d) we obtain after a small calculation for the matrix A the equation

$$A_{tx}-\{A_{xx}, b\}+[A_x, [A, b]]=0. \tag{4.10}$$

It also appears that the system (4.9) is equivalent to the system (4.9b), (4.9c) and (4.10).

If A satisfies equation (4.10) then the following system of equations

$$\begin{pmatrix} y \\ z \end{pmatrix}_x=\left(\begin{array}{c|c} A & I \\ \hline -A^2-\lambda^2 I & -A \end{array}\right)\begin{pmatrix} y \\ z \end{pmatrix}, \qquad \begin{pmatrix} y \\ z \end{pmatrix}_t=\left(\begin{array}{c|c} \{A, b\} & 2b \\ \hline c & -\{A, b\} \end{array}\right)\begin{pmatrix} y \\ z \end{pmatrix} \tag{4.11}$$

with $c=-A_t+\{A_x, b\}-\{A^2, b\}-2\lambda^2 b$ is completely integrable and hence defines a prolongation of equation (4.10)

Following Martini (1983) one may substitute $A=-W+\gamma$, $b=\beta$ and $[\gamma, \beta]=\alpha$ where α, β, γ are constant 2×2 matrices, and we get the so-called Boomeron equation

$$W_{tx}=[\alpha, W_x]+\{W_{xx}, \beta\}+[W_x, [W, \beta]]. \tag{4.12}$$

4.3. Quadratic prolongation and Bäcklund transformation

From (4.11) we can derive at least locally a quadratic prolongation by substituting $\tilde{y}=zy^{-1}$ which gives

$$\begin{aligned} \tilde{y}_x &= -(A+\tilde{y})^2-\lambda^2 I \\ \tilde{y}_t &= -A_t+\{A_x, b\}-\{A^2, b\}-2\lambda^2 b-\{\{A, b\}, \tilde{y}\}-2\tilde{y}b\tilde{y}. \end{aligned} \tag{4.13}$$

Equation (4.13) can be transformed into a more convenient form by substituting

$\tilde{\tilde{y}} = \tilde{y} + A$, which gives, after omitting the double tilde

$$\begin{aligned} y_x &= -y^2 - \lambda^2 I + A_x \\ y_t &= \{A_x - \lambda^2 I, b\} + [[A, b], y] - 2yby. \end{aligned} \tag{4.14}$$

As in the case of the matrix KdV equation, we could try to eliminate A and A_x from (4.14), but since A is not explicitly given, this would become very tedious. We therefore try the following 'ansatz': suppose the transformation

$$y \to -y, \qquad A \to A'$$

leaves equations (4.10) and (4.14) invariant, then it follows from (4.14) that

$$y_x = -y^2 - \lambda^2 I + A_x$$

and

$$-y_x = -y^2 - \lambda^2 I + A'_x$$

or

$$A'_x - A_x = -2y_x.$$

Now, if we put $A' = A - 2y$ we find after a very straightforward calculation the following result.

Theorem 5. The set of equations

$$\begin{aligned} y_x &= -y^2 - \lambda^2 I + A_x \\ y_t &= \{A_x - \lambda^2 I, b\} + [[A, b], y] - 2yby \end{aligned} \tag{4.15}$$

$$A_{tx} - \{A_{xx}, b\} + [A_x, [A, b]] = 0 \tag{4.10}$$

is invariant under the transformation

$$y' = -y, \qquad A' = A - 2y.$$

So we have found a Bäcklund transformation

$$A' = A - 2y \tag{4.16}$$

for equation (4.10), and by substituting $A = -W + \gamma$, $A' = -W' + \gamma$ a Bäcklund transformation

$$W' = W + 2y \tag{4.17}$$

for the Boomeron equation.

4.4. *Construction of a solution for the Boomeron equation*

Since $W \equiv 0$ is a solution of (4.10), we find a new solution of the Boomeron equation by using the Bäcklund transformation (4.17) and equations (4.15), $W' = 2y$ where y satisfies

$$\begin{aligned} y_x &= -y^2 - \lambda^2 I \\ y_t &= -2\lambda^2 b + [[\gamma, b], y] - 2yby \end{aligned}$$

or, with $b = \beta$ and $[\gamma, \beta] = \alpha$

$$y_x = -y^2 - \lambda^2 I, \qquad y_t = -2\lambda^2\beta + [\alpha, y] - 2y\beta y, \qquad W' = 2y. \tag{4.18}$$

To get (4.18) into the form treated by Calogero and Degasparis (1980), we substitute

$$\beta=\sum_{i=1}^{3}\tilde{b}_i\sigma_i,\qquad \alpha=\sum_{i=1}^{3}\tilde{a}_i\sigma_i,\qquad W'=U\sigma_0+\sum_{i=1}^{3}V_i\sigma_i \tag{4.19}$$

where $\sigma_0=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and σ_i, $i=1,2,3$ are the Pauli matrices given by

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

We obtain the result that (4.18) is equivalent to

$$\begin{aligned}&U_x=-\tfrac{1}{2}U^2-\tfrac{1}{2}\boldsymbol{V}\cdot\boldsymbol{V}-2\lambda^2,\qquad \boldsymbol{V}_x=-U\boldsymbol{V},\qquad U_t=-2(\tilde{\boldsymbol{b}}\cdot\boldsymbol{V})U\\ &\boldsymbol{V}_t=-4\lambda^2\tilde{\boldsymbol{b}}+2\mathrm{i}(\tilde{\boldsymbol{a}}\times\boldsymbol{V})-U^2\tilde{\boldsymbol{b}}-(\tilde{\boldsymbol{b}}\cdot\boldsymbol{V})\boldsymbol{V}+\boldsymbol{V}\times(\tilde{\boldsymbol{b}}\times\boldsymbol{V})\end{aligned} \tag{4.20}$$

where $\boldsymbol{V}=(V_1, V_2, V_3)$, $\tilde{\boldsymbol{a}}=(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3)$ and $\tilde{\boldsymbol{b}}=(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3)$. We set $\boldsymbol{a}=2\mathrm{i}\tilde{\boldsymbol{a}}$, $\boldsymbol{b}=2\tilde{\boldsymbol{b}}$, and using the relation $\boldsymbol{V}\times(\boldsymbol{b}\times\boldsymbol{V})=(\boldsymbol{V}\cdot\boldsymbol{V})\boldsymbol{b}-(\boldsymbol{V}\cdot\boldsymbol{b})\boldsymbol{V}$ we find from (4.20)

$$U_x=-\tfrac{1}{2}U^2-\tfrac{1}{2}\boldsymbol{V}\cdot\boldsymbol{V}-2\lambda^2,\qquad \boldsymbol{V}_x=-U\boldsymbol{V}, \tag{4.21a, b}$$

$$U_t=-\boldsymbol{b}\cdot\boldsymbol{V}U,\qquad \boldsymbol{V}_t=U_x\boldsymbol{b}+(\boldsymbol{a}\times\boldsymbol{V})+\boldsymbol{V}\times(\boldsymbol{b}\times\boldsymbol{V}). \tag{4.21c, d}$$

It follows from (4.21b) that we may take $\boldsymbol{V}=\mu(x,t)\boldsymbol{n}(t)$, where $\mu(x,t)$ is a scalar function and $\boldsymbol{n}(t)$ a vector independent of x satisfying $\boldsymbol{n}\cdot\boldsymbol{n}=1$.

We can satisfy (4.21a) and (4.21b) by choosing

$$\mu=U+2\mathrm{i}\lambda$$

so that U satisfies

$$U_x=-U^2-2\mathrm{i}\lambda U. \tag{4.22}$$

Further it follows from (4.21c) that

$$U_t=-\boldsymbol{b}\cdot\boldsymbol{n}(U+2\mathrm{i}\lambda)U \tag{4.23}$$

and hence $U_t=\boldsymbol{b}\cdot\boldsymbol{n}U_x$, therefore $U=U(x-\xi(t))$ with $\xi(t)$ satisfying

$$\xi_t=-\boldsymbol{b}\cdot\boldsymbol{n}. \tag{4.24}$$

We put $\eta=x-\xi(t)$ and it follows from (4.22) that

$$(U+\mathrm{i}\lambda)_\eta=-(U+\mathrm{i}\lambda)^2-\lambda^2 \tag{4.25}$$

and hence

$$U(x,t)=-p(1-\tanh p(x-\xi(t))) \tag{4.26}$$

with $p=+\mathrm{i}\lambda$; this is the 'soliton' solution as found by Calogero and Degasparis (1980).

The only unknown to be solved is the vector $\boldsymbol{n}(t)$. The differential equation to be satisfied by this vector follows readily from the equations (4.21c) and (4.21d). Using the relations $\boldsymbol{V}=(U+2p)\boldsymbol{n}$ and $\boldsymbol{n}\times(\boldsymbol{b}\times\boldsymbol{n})=(\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{b}-(\boldsymbol{n}\cdot\boldsymbol{b})\boldsymbol{n}$ we obtain

$$\boldsymbol{n}_t=\boldsymbol{a}\times n+2p(\boldsymbol{b}-(\boldsymbol{n}\cdot\boldsymbol{b})\boldsymbol{n}). \tag{4.27}$$

The equations (4.27) and (4.24) are the same as those given by Calogero and Degasparis (1980). For further reductions we refer the reader to this reference. However, in order to illustrate the term 'Boomeron' we restrict here our attention to the case $\boldsymbol{a}=\tau\boldsymbol{b}$ with

τ non-zero and real valued. The relation (4.27) yields

$$\boldsymbol{n}_t = \tau\boldsymbol{b}\times\boldsymbol{n}+2p(\boldsymbol{b}-(\boldsymbol{n}\cdot\boldsymbol{b})\boldsymbol{n})$$

or

$$\begin{aligned}\boldsymbol{n}_t\cdot\boldsymbol{b} &= \tau(\boldsymbol{b}\times\boldsymbol{n})\cdot\boldsymbol{b}+2p[\boldsymbol{b}\cdot\boldsymbol{b}-(\boldsymbol{n}\cdot\boldsymbol{b})^2]\\ &= -2p(\boldsymbol{n}\cdot\boldsymbol{b})^2+2p\boldsymbol{b}\cdot\boldsymbol{b}.\end{aligned}$$

Putting $\boldsymbol{n}\cdot\boldsymbol{b}=c$ and $\boldsymbol{b}\cdot\boldsymbol{b}=b^2$ we get the simple differential equation

$$\mathrm{d}c/\mathrm{d}t=-2pc^2+2pb^2,$$

and the regular solution reads

$$c(t)=b\tanh[2pb(t-t_0)]$$

and so

$$\boldsymbol{n}(t)\cdot\boldsymbol{b}=b\tanh[2pb(t-t_0)] \tag{4.28}$$

with t_0 an integration constant. This relation describes the so-called polarisation of the solution.

We then obtain with the aid of (4.24) the wave velocity, appearing in the function $U(x, t)$

$$\xi_t=-b\tanh[2pb(t-t_0)]. \tag{4.29}$$

Taking λ purely imaginary with, for example, $\operatorname{Im}\lambda<0$, we have $p>0$ and the wave speed is positive for $t<t_0$ and negative for $t>t_0$, which explains the name of the 'Boomeron' equation.

Finally, we remark that this wave speed assumes asymptotic values $\pm b$ for $t\to\mp\infty$.

5. Conclusion

It has been shown, using the example of a KdV type system of equations and the Boomeron equation, that a simple reformulation of the prolongation condition, in terms of the integrability condition of a linear system of equations, can lead to a very straightforward determination of a prolongation structure, a Bäcklund transformation and hence the construction of solutions.

Although we have only treated the case of 2×2 matrices, the method can easily be generalised to $N\times N$ matrix systems of equations. The use of exterior differential systems is a powerful method for deriving more general prolongation structures, but it has the disadvantage that in the case of systems of nonlinear evolution equations one is forced to make very long and tedious calculations. As to this it may be of importance to use formula manipulation (see Gragert 1981).

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